Logarithmic Structures on Schemes

This is a write-up of my notes for a talk at MIT BAGS on 11/1/5. My talk stole blatantly from the articles [1] and [2]; please look at them for more details. I thank L. Illusie for explaining much of this to me. — J. P.

The Point

A logarithmic structure, or "log.str" for short, is a datum added to a scheme. In the category of schemes with log.strs, a morphism can be smooth without the underlying scheme morphism being smooth, even though the definition is technically almost identical. Notably, the types of singularities observed on toric varieties are allowed. The result is that we can extend many techniques (notably chomological ones) from classically smooth schemes to log smooth schemes.

This note will be very elementary. In it we present the basic definitions and picture, without proof, and at the end we hint at some ideas that build on these. All results are due to J.-M. Fontaine, L. Illusie, and K. Kato, until the final section, where we mention results due to a large number of authors.

Motivation

Let $X/\mathbb{C}$ be a smooth variety of dimension n. Let $D \subset X$ be a reduced normal crossings divisor: on a suitable polydisc $B \subset X$, we have

$$\begin{array}{ccccc} D \cap B & \subset & B & & \\ \wr\| & & \wr\| & & \\ \{z_1 \cdots z_k = 0\} & \subset & \{|z_i| < 1\} & \subset & \mathbb{C}^n. \end{array}$$

Define $\Omega^1_X(\log D)$, the sheaf of differential forms on X with logarithmic poles along D, to be the sheaf of meromorphic 1-forms on X that are holomorphic away from D and on polydisks B as above can be written as

$$\sum_{i \le k} f_i \frac{dz_i}{z_i} + \sum_{i > k} f_i \, dz_i \qquad \text{with all } f_i \text{ holomorphic.}$$

The higher differentials with log poles are defined similarly, or as $\bigwedge^\bullet \Omega^1_X(\log D)$.

A theorem going back to Grothendieck, as part of his algebraic de Rham theorem, and explained and expanded somewhat by Deligne, gives the existence of a canonical isomorphism

$$\mathbb{H}^*(X, \Omega^\bullet_X(\log D)) \cong H^*((X \backslash D)^{\mathrm{an}}, \mathbb{C}).$$

It provides evidence that these constructions capture some meaningful geometric information.

Monoids

We quickly recall some basics concerning monoids.

A *monoid* is a set equipped with an associative, commutative multiplication law with unit. Homomorphisms are multiplicative maps that preserve the unit element.

A monoid M is called *integral* if it obeys the cancellation law "$ab = ac \implies b = c$". There is an initial map $M \to M^{\text{int}}$ among all maps from M to integral monoids; the integral monoid M^{int} is merely M taken modulo the relations making up the cancellation law. We may similarly form a group M^{gp} and a morphism $M \to M^{\text{gp}}$ that is initial among all maps from M to groups; namely, M^{gp} is obtained from M^{int} by adding inverses of all elements. One has maps $M \twoheadrightarrow M^{\text{int}} \hookrightarrow M^{\text{gp}}$, so that M is integral if and only if $M \hookrightarrow M^{\text{gp}}$.

For a ring R, we denote by $R[M]$ the monoid algebra over R with coefficients in M. This is a free R-module on M, with multiplication defined in the evident manner. For example, writing $\mathbb{N}$ for $\mathbb{Z}_{\geq 0}$, we have $R[\mathbb{N}^n] \cong R[x_1, \ldots, x_n]$.

Pre-log.strs and Log.strs

A *pre-log.str* M on a scheme X is a pair (M, α) with M a sheaf of monoids on the étale site $X_{\text{ét}}$, and $\alpha\colon M \to \mathcal{O}_{X_{\text{ét}}}$ a homomorphism of sheaves of monoids.

We will henceforth sloppily confuse "$\mathcal{O}_{X_{\text{ét}}}$" with "$\mathcal{O}_X$."

Here is an elementary example. Let P be any monoid, and let $\beta_0\colon P \to \Gamma(X_{\text{Zar}}, \mathcal{O}_X)$ be a morphism. If we write P_X for the constant sheaf on $X_{\text{ét}}$ with value P, then β_0 naturally extends to a homomorphism $\beta\colon P_X \to \mathcal{O}_X$, thus yielding a pre-log.str.

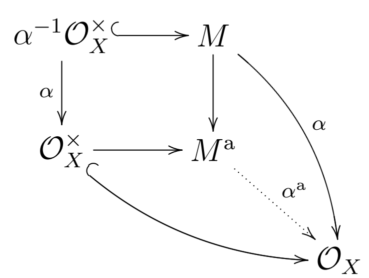
A pre-log.str M is called a *log.str* if α induces an isomorphism $\alpha^{-1}\mathcal{O}_X^\times \xrightarrow{\sim} \mathcal{O}_X^\times$.

The first example is the *trivial* log.str, given by $(\mathcal{O}_X^\times, \text{incl.})$. It is initial among all log.strs on X. The corresponding final object, which is of little use, is $(\mathcal{O}_X, \text{id})$. Another example has $M = \mathcal{O}_X^\times \oplus \mathbb{N}$, and

$$\alpha(a, n) = \begin{cases} a & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

This example shows that α may not be injective.

Just as one associates a sheaf to a presheaf, one can associate a log.str to a pre-log.str. Given a pre-log.str M on X, the *associated log.str* M^{a} is the pushout (fiber coproduct) of the following square, with structure map given by the induced arrow α^{a}:



If $f\colon X \to Y$ is a map of schemes, and N a pre-log.str on Y, we may form the *inverse image pre-log.str* as the inverse image sheaf $f^{-1}N$ together with the composite map $f^{-1}N \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. If N is moreover a log.str, we define the *inverse image log.str* f^*N to be $(f^{-1}N)^{\text{a}}$.

A *(pre-)log scheme* is a scheme equipped with a (pre-)log.str. A *log map* of (pre-)log schemes $(X, M) \to (Y, N)$ is the data of a map $f\colon X \to Y$ of underlying schemes, and an $X_{\text{ét}}$-monoid map $f^{\#}\colon f^{-1}N \to M$. (When X, Y are moreover log schemes, $f^{\#}$ is the same as a map $f^*N \to M$.)

Fine Log.strs

A log map $f\colon X \to Y$ of log schemes is called *strict* if $f^{\#}$ is an isomorphism.

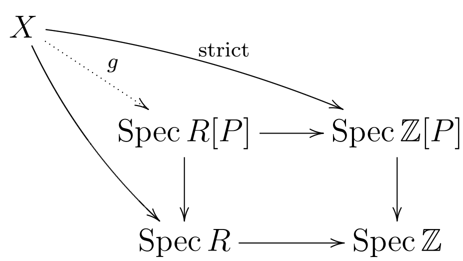
The pre-log.str M is said to be *integral* if it is a sheaf of integral monoids. If M is integral then M^{a} is integral.

We call a log.str M on X *fine* if it is integral and coherent. *Coherence* means that étale-locally on X, there exists an isomorphism $(M, \alpha) \cong (P_X, \beta)^{\mathrm{a}}$, with (P_X, β) as in the elementary example above of a pre-log.str, and P a finitely generated (integral) monoid. Call such a P, together with the map $P_X \to M$ inducing the isomorphism, a *chart* for M, where defined. The existence/data of the chart P can be geometrically rephrased as follows. The inclusion $P \hookrightarrow \mathbb{Z}[P]$ provides a canonical pre-log.str on $\operatorname{Spec} \mathbb{Z}[P]$, and, by association, a canonical log.str. Then the existence/data of the above chart is equivalent to the existence/data of a strict log map $X \to \operatorname{Spec} \mathbb{Z}[P]$.

A *chart* for a log map $f\colon (X, M) \to (Y, N)$ of fine log schemes is a triple of: a chart $P_X \to M$; a chart $Q_Y \to N$; and a map $h\colon Q \to P$ of monoids, all making the appropriate diagram commute.

If M is fine on X, then for every point $x \in X$, and geometric point $\overline{x}$ over x, the quotient stalk $M_{\overline{x}}/\mathcal{O}^{\times}_{X,\overline{x}}$ is a finitely generated integral monoid. One can lift generators étale-locally to a chart $\mathbb{N}^n_X \to \mathcal{O}_X$.

If M is fine on X and X lies over $\operatorname{Spec} R$ in such a way that in the diagram



the induced arrow g is flat, then M is isomorphic to the submonoid of $\mathcal{O}_X$ generated by $\mathcal{O}_X^{\times}$ and P.

We arrive at inclusions of full subcategories as follows:

$$\textbf{FineLogSch} \subset \textbf{CohLogSch} \subset \textbf{LogSch} \subset \textbf{PreLogSch}.$$

The last inclusion has a right adjoint given by $(X, M) \mapsto (X, M^{\mathrm{a}})$. The first inclusions also admits a right adjoint, written $(X, M) \mapsto (X, M)^{\mathrm{int}}$, but be warned: $(X, M)^{\mathrm{int}} \neq$ "(X, M^{int})"; the underlying scheme might change.

Each category has finite inverse limits, including fiber products. Again, a warning: unlike in the other three categories, the limits in **FineLogSch** may result in different underlying schemes than in **Sch**.

Big Examples

(0) Let $i\colon D \hookrightarrow X$ be a closed immersion of schemes. Denoting $j\colon U = X\backslash D \hookrightarrow X$ for the inclusion of the complement, and $\mathcal{O}_U^\times$ for the trivial log.str on U, we put $M_D = j_*\mathcal{O}_U^\times \cap \mathcal{O}_X$. Then M_D is a log.str on X.

(I) Suppose that, in the setup of (0) above, X is of finite type over a field k, and D is reduced normal crossings divisor. This means that, étale-locally on X, the inclusion i is an étale pullback of the inclusion $\{x_1x_2\cdots x_k = 0\} \hookrightarrow \mathbb{A}^n$. Then M_D is fine, and over such an étale neighborhood, writing $\widetilde{x}_i$ for the pullback of x_i to X, we have the chart

$$\begin{array}{ccc} \mathbb{N}_X^k & \longrightarrow & \mathcal{O}_X \\ \uplus & & \uplus \\ (n_i) & \longmapsto & \prod \widetilde{x}_i^{n_i}. \end{array}$$

(II) Let A be a DVR and $S = \operatorname{Spec} A$. The *canonical* log.str on S is M_D with $D = (s)$ as in (0), where $s \in S$ is the closed point; it is fine. Given an S-scheme X of finite type, we similarly equip it with the log.str M_{X_s}, writing X_s for the special fiber of X. The structure map $X \to S$ naturally extends to a log map. We say that X has *(strict) semistable reduction* if X is regular and, étale-locally on X, the map of schemes $X \to S$ factors as

$$X \xrightarrow{\text{ét}} \operatorname{Spec} A[x_1, x_2, \ldots, x_n]/(x_1x_2\cdots x_d - \pi) \to S,$$

for some uniformizer $\pi \in A$. In this case, the log.str on X is fine, and étale-locally has a chart of the form

$$\begin{array}{ccc} Q = \mathbb{N} & \longrightarrow & \mathcal{O}_S \\ \uplus & & \uplus \\ 1 & \longmapsto & \pi \end{array} \qquad \begin{array}{ccc} P = \mathbb{N}^k & \longrightarrow & \mathcal{O}_X \\ \uplus & & \uplus \\ (n_i) & \longmapsto & \prod \widetilde{x}_i^{n_i} \end{array} \qquad \begin{array}{ccc} Q & \xrightarrow{h} & P \\ \uplus & & \uplus \\ 1 & \longmapsto & (1,1,\ldots,1). \end{array}$$

The appearance of "étale-localness" in the definitions of "reduced normal crossing divisor" and "semistable reduction" are the reasons for the use of the étale topology in the definition of a log.str.

Log Differentials and Smoothness

Given a log map $f\colon (X, M) \to (Y, N)$ of pre-log schemes, define the *sheaf of relative differentials with log poles* $\Omega^1_{X/Y}(\log M/N)$, usually abbreviated $\omega^1_{X/Y}$, to be the sheaf on $X_{\text{ét}}$ whose local sections are generated by

$$\begin{array}{ccc} \Omega^1_{X/Y} & \oplus & (\mathcal{O}_X \otimes_{\mathbb{Z}} M^{\text{gp}}) \\ \uplus & & \uplus \\ \varphi & + & a \cdot d\log m, \end{array}$$

modulo the relations

$$\alpha(m) d\log m = d(\alpha(m)), \quad m \in M^{\text{gp}}, \qquad \text{and} \qquad d\log n = 0, \qquad n \in \operatorname{img} f^{-1}N^{\text{gp}}.$$

We define the complex of higher differentials by setting $\omega^\bullet_{X/Y} = \left(\bigwedge^\bullet_{\mathcal{O}_X} \omega^1_{X/Y}, d\right)$, with d extended uniquely to a derivation satisfying $d(d\log m) = 0$.

Since $\Omega^1_{X/Y}(\log M/N) = \Omega^1_{X/Y}(\log M^{\mathrm{a}}/N) = \Omega^1_{X/Y}(\log M^{\mathrm{a}}/N^{\mathrm{a}})$, we see that only true log schemes (and not just pre-log schemes) are relevant here. If M, N are fine, and X is noetherian and of finite type over Y, then $\omega^1_{X/Y}$ is coherent on X.

It is easy to compute $\omega^1_{X/Y}$ in the big examples (I), (II) above. In (I), equipping $\operatorname{Spec} k$ with the trivial log.str, and working étale-locally as above, we have

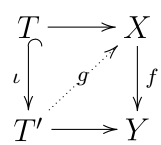
$$\Omega^1_{X/k}(\log M_D) \cong \bigoplus_{i \le k} \mathcal{O}_X \, d\log \widetilde{x}_i \oplus \bigoplus_{i > k} \mathcal{O}_X \, d\widetilde{x}_i.$$

In (II), similarly, we have

$$\Omega^1_{X/S}(\log M/N) \cong \frac{\bigoplus_{i \le k} \mathcal{O}_X \, d\log \widetilde{x}_i}{\mathcal{O}_X \, d\log \widetilde{\pi}} \oplus \bigoplus_{i > k} \mathcal{O}_X \, d\widetilde{x}_i.$$

To define smoothness, we begin by defining a map $\iota\colon X \hookrightarrow Y$ in **LogSch** to be a *strict closed immersion* if it is a closed immersion of underlying schemes that is strict as a log map (so that $\iota^\#$ is an isomorphism).

A map $f\colon X \to Y$ in **FineLogSch** is called *log étale* (resp. *log smooth*) if its underlying map of schemes is of locally finite presentation, and for every commuatative diagram of fine log schemes

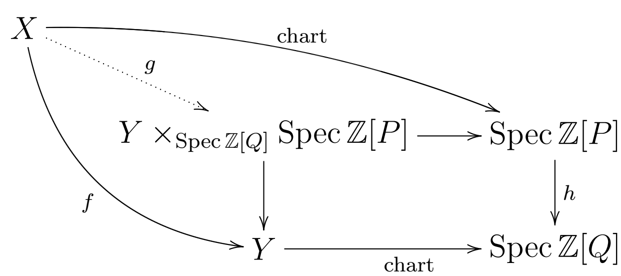


with ι a strict closed immersion, and T defined by a nilpotent ideal in $\mathcal{O}_{T'}$, there exists a unique log map g (resp. étale-locally on T' there exists a log map g) making the diagram commute.

Essentially, we used the log analogue of the old-fashioned "formal smoothness" criterion of smoothness. Classically, there is also the criterion that étale-locally on X, the morphism $X \to Y$ factors as $X \xrightarrow{g} \mathbb{A}^n_Y \to Y$ with g étale. Here is its analogue in the log setting.

A fine log map $f\colon X \to Y$ is log smooth (resp. log étale) if and only if étale-locally on X and Y, f admits a chart $(P \to \mathcal{O}_X, Q \to \mathcal{O}_Y, h\colon Q \to P)$ such that

- h^{gp} is injective, and $(\operatorname{coker} h^{\mathrm{gp}})^{\mathrm{tors}}$ (resp. $\operatorname{coker} h^{\mathrm{gp}}$) is finite of order invertible on X, and
- the induced arrow g is étale in the following fiber product diagram.



Moreover, $\Omega^1_{X/Y}(\log M/N) \cong \mathcal{O}_X \otimes_{\mathbb{Z}} \operatorname{coker} h^{\mathrm{gp}}$, which is clearly locally free of finite rank (resp. vanishes).

Log smoothness is preserved under composition and fine base change. The analogue of the "differential characterization of infinitesimal liftings" works fine, using $\omega^1_{X/Y}$ in place of $\Omega^1_{X/Y}$.

We conclude this section with a list of examples of log smooth maps.

If X and Y are schemes equipped with their trivial log.strs, then a map $f: X \to Y$ is log smooth if and only if it is classically smooth.

If $A \subseteq A'$ is a finite extension of DVRs, and $S^{(\prime)} := \operatorname{Spec} A^{(\prime)}$ with their canonical log.strs as in (II), then the natural extension of $S' \to S$ to a log map is log smooth (equivlaently, log étale) if and only if $\operatorname{Frac} A' / \operatorname{Frac} A$ is tamely ramified.

The log map $X \to S$ of (II), with X having semistable reduction, is log smooth.

Combining the preceding two examples, the fine base change $X_{S'} \to S'$ is log smooth, even though $X_{S'}$ does *not* necessarily have semistable reduction if $\operatorname{Frac} A' / \operatorname{Frac} A$ has ramification index > 1.

In the situation of (I), equip $\operatorname{Spec} k$ with the trivial log.str, and extend $X \to \operatorname{Spec} k$ to a log map in the unique way. If X is regular, then X/k is log smooth. More generally, if (X, M) is a fine log scheme of finite type over k, then X/k is log smooth if and only if either of these two equivalent conditions hold:

- Étale-locally on X, there exists a finitely generated integral monoid P and an étale morphism of underlying schemes $X \to \operatorname{Spec} k[P]$ such that $M \cong P \cdot \mathcal{O}_X^\times \subseteq \mathcal{O}_X$ and $(P^{\mathrm{gp}})^{\mathrm{tors}}$ is finite of order invertible in k.

- X is a "toroidal embedding", étale-locally given by the open immersion $X \times_{\operatorname{Spec} k[P]} \operatorname{Spec} k[P^{\mathrm{gp}}] \subset X$.

(In the latter situation, classically, one assumes that $(P^{\mathrm{gp}})^{\mathrm{tors}} = 0$.)

Geometry (and Cohomology) of Log Schemes

In this last section, we mention a scattering of topics beyond the basic definitions that begin to give a feel for the geometric properties of log schemes. There is *much more* to the subject than is covered here, and many authors' work will be used without reference. These ommissions are entirely due to my lack of time to learn more.

Given a fine log scheme (X, M), put $\overline{M^{\mathrm{gp}}} = M^{\mathrm{gp}}/\mathcal{O}_X^\times$, and define subsets of X by $X_{\mathrm{triv}} = \{x \in X \mid \overline{M^{\mathrm{gp}}}_{\overline{x}} = 1\}$ and $X_i = \{x \in X \mid \operatorname{rank}_{\mathbb{Z}} \overline{M^{\mathrm{gp}}}_{\overline{x}} \geq i\}$.

The subset X_{triv} is the maximal open subset on which M "is" the trivial log.str; if X is regular and of finite type over a field k, then X_{triv} is open, and its complement is a divisor from which M is obtained as in (I).

The subsets X_i generalize the stratification of a toric variety by the closures of its orbits under the torus action. The parallel to toric geometry runs deep: resuolution of singularities for toric varieties can be adapted to apply here, using log blowups.

Log.strs are classified by a sort of "toric quotient" stack, constructed by M. Olsson. Once set up in this framework, many definitions are easier to state, and many concepts are easier to wield.

Define the *polar log.str* on $\operatorname{Spec}\mathbb{C}$ by setting $M_{\text{polar}} = \mathbb{R}_{\geq 0} \times S^1$, thinking of S^1 as the unit complex numbers, and using

$$\begin{array}{rcl} \alpha\colon \mathbb{R}_{\geq 0} \times S^1 & \longrightarrow & \mathbb{C}, \\ \cup\!\!\!\!\!\!\quad & & \cup \\ (r,s) & \longmapsto & rs. \end{array}$$

Given X in **FineLogSch** that is smooth and of finite type over $(\operatorname{Spec}\mathbb{C}, \text{trivial})$, define

$$X^{\log} := \operatorname{Hom}_{\mathbf{LogSch}}((\operatorname{Spec}\mathbb{C}, M_{\text{polar}}), X).$$

There is a canonical topology on the set $X^{\log}$ making it a C^∞-smooth manifold with boundary/corners. We think of it as a topological log blowup of X.

By taking a log map $f \in X^{\log}$ and "forgetting $f^\#$", we get a point in X^{an}; the resulting map $\tau_X\colon X^{\log} \to X^{\text{an}}$ is continuous, proper, and an isomorphism on X_{triv}. More precisely, for a point $x \in X^{\text{an}}$, the fiber $\tau_X^{-1}(x)$ is a product of $r(x)$ copies of the circle S^1, where $r(x) = \operatorname{rank}_{\mathbb{Z}} \overline{M^{\text{gp}}}_{\overline{x}}$.

If M is of the form M_D as in (I), then one can think of $X^{\log}$ inflating the underlying subspace of D into a "generalized tube". The composite map

$$(X \backslash D)^{\text{an}} \xleftarrow{\sim} (X\backslash D)^{\log} \hookrightarrow X^{\log}$$

is a homotopy isomorphism of topological spaces. (Go ahead and try this when $X = \mathbb{A}^1_{\mathbb{C}}$, with D the divisor of the origin!) This homotopy equivalence figures into the algebraic de Rham theorem mentioned in the motivational section.

All cohomology theories behave nicely with respect to classically smooth maps, and most of them admit "log" versions that are nice with respect to log smooth maps. This was done first for log-crystalline cohomology, but other standard ones soon followed, including the log-Betti, log-étale, and log-de Rham cohomologies.

These have proved *very* useful for studying varieties that are classically nonsmooth but are log smooth, where old-fashioned cohomology theories aren't applicable. We mention as examples T. Tsuji's proof of Fontaine's conjecture C_{st}, and O. Gabber's recent work on finiteness and Euler characteristics in classical étale cohomology.

References

[1] K. Kato, "Logarithmic Structures of Fontaine–Illusie". Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 191–224, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[2] L. Illusie, "An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic etale cohomology." Cohomologies p-adiques et applications arithmetiques, II. Asterisque No. 279, (2002), 271–322.